Elementary Number Theory.

Hints of Problems 2-2. *by Jonathan Tsai*

3.(a) Use **Division Algorithm** to show that :
The square of any integer is either of the form $3K$ or $3K+1$.

Proof.
(1) $0^2 = 3 \cdot 0$ is of the form $3K$.
(2) Let $n \in \mathbb{N}$.
Then by **Division Algorithm**, $\exists!\ q,\ r \in \mathbb{Z}$ such that : $n = 3q + r$ and $0 \leqslant r < 3$.
Case 1 : If $n = 3q$, $n^2 = (3q)^2 = 9q^2 = 3(3q^2) = 3K$ for some $K \in \mathbb{Z}$.
Case 2 : If $n = 3q+1$, $n^2 = (3q+1)^2 = 9q^2 + 6q + 1 = 3(3q^2+2q) + 1 = 3K + 1$ for some $K \in \mathbb{Z}$.
Case 3 : If $n = 3q+2$, $n^2 = (3q+2)^2 = 9q^2 + 12q + 4 = 3(3q^2+4q+1) + 1 = 3K+1$ for some $K \in \mathbb{Z}$.
By (1) and (2) $\Rightarrow n^2$ is either of the form $3K$ or $3K+1$, $\forall n \in \mathbb{N} \cup \{0\}$.
$\Rightarrow n^2$ is either of the form $3K$ or $3K+1$, $\forall n \in \mathbb{Z}$. □

4. Show that $3a^2 - 1$ is not a perfect square, $\forall a \in \mathbb{Z}$.

Proof.
$\because 3a^2 - 1 = 3(a^2-1) + 2 = 3K + 2$ for some $K \in \mathbb{Z}$, $\forall a \in \mathbb{Z}$.
$\therefore$ By the uniqueness of **Division Algorithm** :
$3a^2 - 1$ is neither in the form of $3K$ nor $3K+1$, $\forall a \in \mathbb{Z}$.
$\Rightarrow 3a^2 - 1$ is not a perfect square, $\forall a \in \mathbb{Z}$. (by **3.(a)**) □

10. Show that $6 | n(7n^2+5)$, $\forall n \in \mathbb{N}$.

Proof. **Method 1:**
Let $n \in \mathbb{N}$.
Then by **Division Algorithm**, $\exists!\ q,\ r \in \mathbb{Z}$ such that : $n = 3q + r$ and $0 \leqslant r < 3$.
Case 1 : If $n = 3q$, $n(7n^2+5) = (3q)(7(3q)^2+5) = 3((3q)^3 + 5q) = 3K$ for some $K \in \mathbb{Z}$.
Case 2 : If $n = 3q+1$,
$n(7n^2+5) = (3q+1)(7(3q+1)^2+5) = 3(7q(3q+1)^2+5) + (7(3q+1)^2+5)$
$= 3K_1 + 7(9q^2+6q+1)+5 = 3K_1 + 7(9q^2+6q)+7+5 = 3K_1 + 3[7(3q^2+2q)+4] = 3K_1 + 3K_2 = 3K$ for some $K_1,\ K_2,\ K \in \mathbb{Z}$.
Case 3 : If $n = 3q+2$,
$n(7n^2+5) = (3q+2)(7(3q+2)^2+5) = 3(7q(3q+2)^2+5) + 2(7(3q+2)^2+5)$
$= 3K_1 + 14(9q^2+12q+4)+10 = 3K_1 + 14(9q^2+12q) + 56 + 10 = 3K_1 + 3(14(3q^2+4q)+22)$
$= 3K_1 + 3K_2 = 3K$ for some $K_1,\ K_2,\ K \in \mathbb{Z}$.

By above, we have: $3|n(7n^2+5)$, $\forall n \in \mathbb{N}$. ——(*)

On the other hand,

If n is **even**, then clearly that $n(7n^2+5)$ is even.

If n is **odd**, then $7n^2$ is odd and hence $n(7n^2+5)$ is even.

Therefore $2|n(7n^2+5)$, $\forall n \in \mathbb{N}$. ——(**)

By (*) and (**), we get: $6|n(7n^2+5)$, $\forall n \in \mathbb{N}$. $\square$

Proof. **Method 2:**

Let $n \in \mathbb{N}$. Then:

$n(7n^2+5) = n(6n^2+6+n^2-1) = 6(n^3+n)+n(n^2-1) = 6K_1+(n+1)n(n-1)$ for some $K_1 \in \mathbb{N}$.

$\because \binom{n+1}{3} \in \mathbb{N}, \forall n \in \mathbb{N}. \therefore \frac{(n+1)n(n-1)}{3!} \in \mathbb{N}$.

$\Rightarrow (n+1)n(n-1) = 6K_2$ for some $K_2 \in \mathbb{N}$.

$\Rightarrow n(7n^2+5) = 6K_1 + 6K_2 = 6K$ for some $K \in \mathbb{N}$.

$\Rightarrow 6|n(7n^2+5)$, $\forall n \in \mathbb{N}$. $\square$